Equal Sums of Biquadrates

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Solutions of the Diophantine equation $A^4 + B^4 = C^4 + D^4$ in least integers have been obtained by several authors [1]–[4]. The term *primitive* denotes a solution for which unity is the greatest common divisor of all the numbers A, B, C, D. A CDC 3200 computer program was written to search exhaustively for primitives, yielding the 31 solutions listed in Table I. The range covered is $A^4 + B^4 < 7.885 \times 10^{15}$. The first six solutions were identified in [3] and the seventh is cited in [1].

Euler [1] gave a two-parameter algebraic solution which can be written

$$A = f(x, y)B = f(y, -x)C = f(-x, y)D = f(y, x),$$

TABLE I

Primitive Solutions of $N = A^4 + B^4 = C^4 + D^4$

i	N_i	A	B	C	D
1	635,318,657	158	59	134	133
2	3,262,811,042	239	7	227	157
3	8,657,437,697	292	193	257	256
4	68,899,596,497	502	271	497	298
5	86,409,838,577	542	103	514	359
6	160,961,094,577	631	222	558	503
7	2,094,447,251,857	1203	76	1176	653
8	4,231,525,221,377	1381	878	1342	997
9	26,033,514,998,417	2189	1324	1997	1784
10	37,860,330,087,137	2461	1042	2141	2026
11	61,206,381,799,697	2797	248	2524	2131
12	76,773,963,505,537	2949	1034	2854	1797
13	109,737,827,061,041	3190	1577	2986	2345
14	155,974,778,565,937	3494	1623	3351	2338
15	156,700,232,476,402	3537	661	3147	2767
16	621,194,785,437,217	4883	2694	4397	3966
17	652,057,426,144,337	5053	604	5048	1283
18	680,914,892,583,617	4849	3364	4303	4288
19	1,438,141,494,155,441	6140	2027	5461	4840
20	1,919,423,464,573,697	6619	274	5942	5093
21	2,089,568,089,060,657	6761	498	6057	5222
22	2,105,144,161,376,801	6730	2707	6701	3070
23	3,263,864,585,622,562	7557	1259	7269	4661
24	4,063,780,581,008,977	7604	5181	7037	6336
25	6,315,669,699,408,737	8912	1657	7559	7432
26	6,884,827,518,602,786	9109	635	9065	3391
27	7,191,538,859,126,257	9018	4903	8409	6842
28	7,331,928,977,565,937	9253	1104	8972	5403
29	7,362,748,995,747,617	9043	5098	8531	6742
30	7,446,891,977,980,337	9289	1142	9097	4946
31	7,532,132,844,821,777	9316	173	9197	4408

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where $f(x, y) = 2x^7 - x^6y + 20x^5y^2 + 17x^4y^3 + 2x^3y^4 + 17x^2y^5 + 8xy^6 - y^7$. The primitives corresponding to $i = 1, 7$ and 14 of Table I are special cases of this solution for the arguments $(x, y) = (3, 1), (2, 1)$, and $(5, 1)$ respectively.

The computer program generated all values of $N = A^4 + B^4$ in ascending order by controlling the advance of a series of pairs of values A, B while monitoring N for coincidences. To advance from a given starting value of N, all integers A for which $N/2 \leqq A^4 \leqq N$ were considered; for each A a corresponding B was chosen as the largest integer in the range $0 \leqq B \leqq A$ for which $A^4 + B^4 \leqq N$. Then the smallest value $A_1^4 + B_1^4$ in the set was found and B_1 was advanced if $B_1 < A_1$, the lower limit on A was advanced if $B_1 = A_1$, and the upper limit on A was advanced if $B_1 = 0$.

A similar computer program generated sums of three biquadrates $A^4 + B^4 + C^4$ in ascending order and found the least triple coincidence to be

$$811{,}538 = 29^4 + 17^4 + 12^4 = 28^4 + 21^4 + 7^4 = 27^4 + 23^4 + 4^4.$$

It was discovered quite by chance (using a computer program which decomposes numbers into sums of biquadrates by trial) that for the N_i of Table I

$$N_1 + 1 = 635{,}318{,}658 = 159^4 + 58^4 + 1^4 = 134^4 + 133^4 + 1^4 = 154^4 + 83^4 + 71^4$$

is the sum of three biquadrates in three distinct ways, and that

$$N_3 + 1 = 8{,}657{,}437{,}698$$

is the sum of three biquadrates in five distinct ways, namely

$$(296{,}157{,}139)^4 = (293{,}184{,}109)^4 = (292{,}193{,}1)^4 = (271{,}239{,}32)^4 = (257{,}256{,}1)^4.$$

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2. C. S. Ogilvy, *Tomorrow's Math*, Oxford, 1962, p. 94.
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